

Singularities of the Isospectral Hilbert Scheme

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Abstract

We study the singularities of the isospectral Hilbert scheme B^n of n points over a smooth algebraic surface and we prove that they are canonical if $n \leq 5$, log-canonical if $n \leq 7$ and not log-canonical if $n \geq 9$. We describe as well two explicit log-resolutions of B^3 , one crepant and the other \mathfrak{S}_3 -equivariant.

Introduction

The aim of this work is the study of the singularities of the isospectral Hilbert scheme of n points over a smooth complex algebraic surface. If X is such a surface, the isospectral Hilbert scheme B^n can be defined as the blow-up of the product variety X^n along the big diagonal Δ_n . The isospectral Hilbert scheme has been introduced by Haiman in his works [Hai99] and [Hai01] on Macdonald polynomials; it was proven in [Hai01] that B^n is normal, Cohen-Macaulay and Gorenstein. Haiman himself asked in [Hai04, Section 1] whether the Rees algebra $\oplus_{i \geq 0} \mathcal{I}_{\Delta_n}^i$ were of F -rational type; this would be equivalent of $\text{Spec } \oplus_{i \geq 0} \mathcal{I}_{\Delta_n}^i$ having rational singularities [Smi97, Har98, MS97, ST08] and would imply [Hyr99, Proposition 1.2] that $B^n = \text{Proj}(\oplus_{i \geq 0} \mathcal{I}_{\Delta_n}^i)$ would have rational or, equivalently, canonical singularities. It is an open problem whether B^n has canonical or log-canonical singularities. In this work we partially answer these questions.

Apart from being interesting in its own, the investigation of the singularities of B^n is in tight relation with a number of interesting problems. The first and more immediate — which is one of the main motivations of this work — is the potential application to vanishing theorems, since sufficiently good singularities would allow the use of Kawamata-Viehweg or Kodaira vanishing over B^n ; an example of this use already appeared in [Sca15, Section 5.2].

A second source of interest, which also offers an effective way to address the problem, is the link with the study of log-canonical thresholds of subspace arrangements. Since B^n is the blow-up of the big diagonal in X^n , it turns out that the scheme B^n — or, in other words, the pair (B^n, \emptyset) — has exactly the same kind of singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$. Now, one can determine the kind of singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$ by studying its log-canonical threshold at each point. Since this problem is now local in nature, one can take X as the affine plane \mathbb{C}^2 : in this case the big diagonal Δ_n can be thought as a subspace arrangement. This problem is similar with that of finding log-canonical thresholds of hyperplane arrangements, already studied and solved in [Mus06]. On the other hand, there are not many examples in literature of computations of log-canonical thresholds of arrangements of subspaces of higher codimension: an exception is the study of configurations of lines through the origin in \mathbb{C}^3 by Teitler [Tei07]. An important part of his work deals with the understanding of the embedded components that appear when pulling back the ideal of the configuration of lines to the blow-up of the origin in \mathbb{C}^3 ; the presence of embedded components is the main difficulty that hinders an explicit log-resolution of the ideal of the configuration.

The case of the pair $(X^n, \mathcal{I}_{\Delta_n})$ — for $X = \mathbb{C}^2$ — is similar because we deal with an arrangement of codimension 2 subspaces Δ_n in \mathbb{C}^{2n} , but the complexity of the problem grows very rapidly with n . However, for $X = \mathbb{C}^2$, Haiman gave a precise description of a set of generators for the ideal \mathcal{I}_{Δ_n} , from which we can deduce the order of the ideal \mathcal{I}_{Δ_n} at each point. As a consequence, we can establish the upper bound (proposition 2.10)

$$\text{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq \frac{2n-2}{d_n}$$

for the log-canonical threshold of the pair $(X^n, \mathcal{I}_{\Delta_n})$. Here d_n is the natural number defined in remark 2.7. We actually believe that the above inequality is in fact an equality (Conjecture 1). This would imply that the singularities of B^n are canonical if and only if $n \leq 7$, log-canonical if $n \leq 8$ and not log-canonical if $n \geq 9$ (Conjecture 2). We can actually prove — and this is the main result of this work —

Theorem 2.12. *The singularities of the isospectral Hilbert scheme B^n are canonical if $n \leq 5$ and log-canonical if $n \leq 7$. For $n \geq 9$ they are not log-canonical.*

Not unexpectedly, this problem is in close relation with the geometry of the Hilbert scheme of points as well. Indeed, after a result by Song in [Son14], results about the pair $(X^n, \mathcal{I}_{\Delta_n})$ can be precisely translated into results about the pair $(X^{[n]}, \mathcal{I}_{\partial X^{[n]}})$, where $X^{[n]}$ is the Hilbert scheme of n points over X and $\partial X^{[n]}$ is its boundary. In particular the previous upper bound for $\text{lct}(X^n, \mathcal{I}_{\Delta_n})$ implies the upper bound $\text{lct}(X^{[n]}, \mathcal{I}_{\partial X^{[n]}}) \leq (n-1)/d_n$. The mentioned conjecture on $\text{lct}(X^n, \mathcal{I}_{\Delta_n})$ would imply that the last upper bound is actually an equality.

Finally, the problem of understanding the singularities of the isospectral Hilbert scheme should be a drive to the construction of an explicit \mathfrak{S}_n -equivariant log-resolution of B^n , or — what is equivalent — to an explicit \mathfrak{S}_n -equivariant log-resolution $f : Y \longrightarrow X^n$ of the pair $(X^n, \mathcal{I}_{\Delta_n})$. This would be a deep and important result on many levels. Firstly, it would provide another important compactification of the configuration space $F(X, n) := X^n \setminus \Delta_n$ after the celebrated Fulton-MacPherson compactification $X[n]$ (see [FM94]): the latter is not, unfortunately, a log-resolution of the pair $(X^n, \mathcal{I}_{\Delta_n})$, since, when computing the inverse image of the ideal \mathcal{I}_{Δ_n} over $X[n]$ embedded components appear. Hence an explicit \mathfrak{S}_n -equivariant log-resolution of $(X^n, \mathcal{I}_{\Delta_n})$ might be built by further blowing-up the Fulton-MacPherson compactification in order to get rid of these components; however, it is a very difficult problem to track and control the embedded components that arise in this way.

Secondly, supposing that the stabilizers of the \mathfrak{S}_n -action on the resolution Y were trivial, then, passing to the quotient would provide an explicit resolution $\hat{f} : Y/\mathfrak{S}_n \longrightarrow S^n X$ of the symmetric variety. We mention that, in general, no such explicit resolution is known yet. In [Uly02] Ulyanov made a step forward proposing a refinement of the Fulton-MacPherson compactification in a way that the stabilizers of the natural \mathfrak{S}_n -action are abelian, and not just solvable.

Finally, such a resolution $f : Y \longrightarrow X^n$ might be useful for a better understanding of ideal sheaves of subschemes supported in big diagonals of the form $\mathcal{O}(-\lambda\Delta)$, appeared in the work [Sca15].

In the final section of this article we provide two different log-resolutions of the pair $(X^3, \mathcal{I}_{\Delta_3})$, and hence of B^3 : one crepant, the other \mathfrak{S}_3 -equivariant.

We work over the field of complex numbers. By point we always mean a closed point.

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1 Singularities of pairs and log-canonical thresholds

Definition 1.1. [Kol97, Laz04] Let M be an irreducible complex algebraic variety, and \mathfrak{a} an ideal sheaf of \mathcal{O}_M . A *log-resolution* of the pair (M, \mathfrak{a}) is a projective birational map $f : Y \longrightarrow M$ such that Y is nonsingular, the exceptional locus $\text{exc}(f)$ is a divisor, the ideal sheaf $f^{-1}\mathfrak{a} := \mathfrak{a} \cdot \mathcal{O}_Y$ is equal to $\mathcal{O}_Y(-F)$, where F is an effective divisor on Y with the property that $F + \text{exc}(f)$ has simple normal crossing support.

Definition 1.2. Let M be a complex algebraic variety, normal and irreducible; let K_M be its canonical divisor. Suppose that M is \mathbb{Q} -Gorenstein, that is, for some $r \in \mathbb{N}^*$, rK_M is Cartier. Let \mathfrak{a} be an ideal sheaf of \mathcal{O}_M . Consider a log-resolution $f : Y \longrightarrow M$ of the pair (M, \mathfrak{a}) . Then, as \mathbb{Q} -Cartier divisors,

$$K_Y - f^*(K_M) + f^{-1}(\mathfrak{a}) = \sum_i a_i E_i,$$

where E_i are irreducible component of a simple normal crossing divisor and $a_i \in \mathbb{Q}$. We say that the singularities of the pair (M, \mathfrak{a}) are *canonical* if $a_i \geq 0$; *log-canonical* if $a_i \geq -1$.

Definition 1.3. Let M be a smooth algebraic variety and \mathfrak{a} an ideal sheaf of \mathcal{O}_M . Let $c \in \mathbb{Q}$, $c > 0$. Let $f : Y \longrightarrow M$ be a log-resolution of the pair (M, \mathfrak{a}) and let F be the effective Cartier divisor on Y such that $f^{-1}\mathfrak{a} = \mathcal{O}_Y(-F)$. Then the *multiplier ideal sheaf* $\mathcal{J}(c \cdot \mathfrak{a})$ associated to c and \mathfrak{a} is the ideal sheaf of \mathcal{O}_M defined as

$$\mathcal{J}(c \cdot \mathfrak{a}) := f_* \mathcal{O}_Y(K_{Y/M} - [c \cdot F]) ,$$

where $[c \cdot F]$ is the integral part of the \mathbb{Q} -divisor F . The definition just given does not depend on the choice of the log-resolution [Laz04]. For $x \in M$, the *log-canonical threshold* of the pair (M, \mathfrak{a}) at the point x is defined as

$$\mathrm{lct}_x(M, \mathfrak{a}) := \sup\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot \mathfrak{a})_x = \mathcal{O}_{M,x}\} = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot \mathfrak{a})_x \subset \mathfrak{m}_x\} .$$

Define, moreover, $\mathrm{lct}(M, \mathfrak{a}) := \inf_{x \in M} \mathrm{lct}_x(M, \mathfrak{a})$.

Remark 1.4. In the above definition of $\mathrm{lct}_x(M, \mathfrak{a})$ the inf are actually minima [Laz04, Example 9.3.16].

Proposition 1.5. *Let M be a smooth complex algebraic variety and let \mathfrak{a} be an ideal sheaf of \mathcal{O}_M . Consider the blow-up $g : B := \mathrm{Bl}_{\mathfrak{a}} M \longrightarrow M$ of M along the ideal \mathfrak{a} , with exceptional divisor E . Suppose that B is irreducible, normal and Gorenstein; suppose moreover that $K_B = g^*K_M + \mathcal{O}_B(E)$. Then B has (log-) canonical singularities if and only if the pair (M, \mathfrak{a}) has.*

Proof. Let $h : Y \longrightarrow B$ be a log-resolution of the pair (B, E) . Consider the map $f = g \circ h$. We claim that f is a log-resolution of the pair (M, \mathfrak{a}) . Indeed $\mathrm{exc}(f)$ is divisorial, since f is a birational morphism between smooth varieties. Moreover, set-theoretically, $\mathrm{exc}(f) = \mathrm{exc}(h) \cup h^{-1}\mathrm{exc}(g) = \mathrm{exc}(h) \cup h^{-1}E$, which — since h is a log-resolution of (B, E) — is a divisor with snc support. Hence $\mathrm{exc}(f)$ is a divisor with snc support. Moreover $f^{-1}\mathfrak{a} = h^{-1}g^{-1}\mathfrak{a} = h^{-1}\mathcal{I}_E = \mathcal{O}_B(-h^*E)$ and h^*E is an effective Cartier divisor. Finally, as Cartier divisors, $\mathrm{exc}(f) + h^*E$ coincides with $\mathrm{exc}(h) + 2h^*E$, which has the same support as $\mathrm{exc}(f)$ and hence is a divisor with snc support. Then

$$K_Y - h^*K_B = K_Y - h^*g^*K_M - h^*\mathcal{O}_B(E) = K_Y - f^*K_M + f^{-1}\mathfrak{a}$$

which allows us to conclude. □

2 The isospectral Hilbert scheme

Definition 2.1. Let $n \in \mathbb{N}$, $n \geq 2$. Let X be a smooth complex algebraic surface. Let Δ_n be the big diagonal in X^n , that is, Δ_n is the scheme-theoretic union of pairwise diagonals Δ_{ij} , $1 \leq i < j \leq n$. The *isospectral Hilbert scheme* B^n is the blow up of X^n along the big diagonal Δ_n .

Remark 2.2. It is well known that the isospectral Hilbert scheme B^n is irreducible, normal, Cohen-Macaulay and Gorenstein [Hai01].

2.1 The big diagonal in X^n

As an immediate consequence of proposition 1.5, we have a very precise correspondence between the singularities of the isospectral Hilbert scheme B^n and those of the pair $(X^n, \mathcal{I}_{\Delta_n})$.

Corollary 2.3. *The isospectral Hilbert scheme B^n has (log-) canonical singularities if and only if the pair $(X^n, \mathcal{I}_{\Delta_n})$ has (log-) canonical singularities.*

Remark 2.4. It is well known [Laz04, Example 9.3.16] that a pair (M, \mathfrak{a}) has log-canonical singularities if and only if $\mathrm{lct}(M, \mathfrak{a}) \geq 1$. On the other hand, if M is Gorenstein, then the discrepancies a_i in definition 1.2 are necessarily integers; consequently the pair (M, \mathfrak{a}) is canonical if and only if $\mathrm{lct}(M, \mathfrak{a}) > 1$, that is, if and only if $\mathcal{J}(M, \mathfrak{a}) = \mathcal{O}_M$. Hence we have that the isospectral Hilbert scheme B^n has canonical singularities if and only if $\mathrm{lct}(X^n, \mathcal{I}_{\Delta_n}) > 1$ or, equivalently, if $\mathcal{J}(X^n, \mathcal{I}_{\Delta_n})$ is trivial; the singularities of B^n are log-canonical if and only if $\mathrm{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1$.

Remark 2.5. The log-canonical threshold $\text{lct}_x(M, \mathfrak{a})$ at the point $x \in M$ coincides with the *complex singularity exponent* $c_x(\mathfrak{a})$ of \mathfrak{a} at the point x [DK01], which is an holomorphic invariant. As a consequence, the log-canonical threshold of the pair $(X^n, \mathcal{I}_{\Delta_n})$ for an arbitrary smooth algebraic surface X is equal to the log-canonical threshold of the pair $((\mathbb{C}^2)^n, \mathcal{I}_{\Delta_n})$.

Remark 2.6 (Generators of \mathcal{I}_{Δ_n} for $X = \mathbb{C}^2$). In [Hai01] Haiman finds an explicit set of generators for ideal of the big diagonal Δ_n of $(\mathbb{C}^2)^n$. Write $(\mathbb{C}^2)^n$ as $\text{Spec } \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$. If $\bar{p}, \bar{q} \in \mathbb{N}^n$, denote with $\Delta(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ the \mathfrak{S}_n -anti-invariant regular function

$$\Delta(\bar{p}, \bar{q}, \bar{x}, \bar{y}) := \det(x_i^{p_j} y_i^{q_j})_{ij}$$

in the variables $x_1, \dots, x_n, y_1, \dots, y_n$. If there is no risk of confusion, we will drop the indication of the variables and we will just write it as $\Delta(\bar{p}, \bar{q})$. Haiman proves that homogeneous polynomials of the form $\Delta(\bar{p}, \bar{q})$ generate the ideal \mathcal{I}_{Δ_n} . Of course the function $\Delta(\bar{p}, \bar{q})$ is non identically zero if and only if the points $(p_i, q_i) \in \mathbb{N} \times \mathbb{N}$ are all distinct.

Remark 2.7 (Generators of minimal degree in \mathcal{I}_{Δ_n}). A nonzero homogeneous polynomial of the form $\Delta(\bar{p}, \bar{q})$ is of minimal degree if the set of points $\{(p_i, q_i), i = 1, \dots, n\}$ minimize the weight $\sum_i (p_i + q_i)$. Now for any $n \in \mathbb{N}$ there exist two natural numbers k and h , with $h < k$, uniquely determined by n , such that $n = k(k+1)/2 + h$. The integers k and h explain how to arrange n distinct points (p_i, q_i) in $\mathbb{N} \times \mathbb{N}$ in such a way that the weight $\sum_i (p_i + q_i)$ is the minimum possible: fill in the first antidiagonals in $\mathbb{N} \times \mathbb{N}$, of weight 0 to $k-1$, with $k(k+1)/2$ points of nonnegative integral coordinates and on the antidiagonal of weight k put, in an arbitrary way, h points. Consequently, a generator of minimal degree has degree

$$d_n = \sum_{i=0}^{k-1} i(i+1) + hk = \frac{1}{3}k(k^2 + 3h - 1).$$

Remark 2.8. Consider the diagonal Δ_n inside $(\mathbb{C}^2)^n = \text{Spec } \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$ and consider its ideal $\mathcal{I}_{\Delta_n} \subseteq \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$. We build now a new coordinate system, in the following way. Consider the vector space $(\mathbb{C}^2)^{n-1}$ with coordinates $(z_1, w_1, \dots, z_{n-1}, w_{n-1})$ and \mathbb{C}^2 with coordinates (α, β) . Consider now the isomorphism

$$\varphi : (\mathbb{C}^2)^n \longrightarrow (\mathbb{C}^2)^{n-1} \times \mathbb{C}^2 \quad (2.1)$$

defined by the coordinate change

$$\begin{aligned} z_i &= x_1 - x_{i+1}, & w_i &= y_1 - y_{i+1} & \text{for } i = 1, \dots, n-1 \\ \alpha &= \sum_{i=1}^n x_i, & \beta &= \sum_{i=1}^n y_i. \end{aligned}$$

In the new coordinates the pairwise diagonals in $(\mathbb{C}^2)^n$ are now given by ideals (z_i, w_i) and $(z_i - z_j, w_i - w_j)$, $1 \leq i < j \leq n-1$ and the ideal \mathcal{I}_{Δ_n} is the intersection

$$\mathcal{I}_{\Delta_n} = \cap_{i=1}^{n-1} (z_i, w_i) \bigcap \cap_{1 \leq i < j \leq n-1} (z_i - z_j, w_i - w_j)$$

inside $\mathbb{C}[z_1, w_1, \dots, z_{n-1}, w_{n-1}, \alpha, \beta]$. Since the generators of \mathcal{I}_{Δ_n} are just polynomials in the z_i, w_i , the ideal \mathcal{I}_{Δ_n} is the extension of an ideal $\mathcal{I}_{\tilde{\Delta}_{n-1}} \subseteq \mathbb{C}[z_1, \dots, z_{n-1}, w_1, \dots, w_{n-1}]$, generated by the same elements. In other words, we can write

$$\mathcal{I}_{\Delta_n} \simeq \varphi^*(\mathcal{I}_{\tilde{\Delta}_{n-1}} \boxtimes \mathcal{O}_{\mathbb{C}^2}). \quad (2.2)$$

Consider now the projection $r : (\mathbb{C}^2)^{n-1} \times \mathbb{C}^2 \longrightarrow (\mathbb{C}^2)^{n-1}$. Under the identification φ , the small diagonal $\Delta_{1, \dots, n}$ in $(\mathbb{C}^2)^n$ is the pre-image $r^{-1}(\{0\})$ by r of the origin $\{0\}$ in $(\mathbb{C}^2)^{n-1}$. Consequently, the order of the big diagonal Δ_n along the small diagonal $\Delta_{1, \dots, n}$ coincide with the order of $\tilde{\Delta}_{n-1}$ at the origin: $\text{ord}_{\Delta_{1, \dots, n}} \mathcal{I}_{\Delta_n} = \text{ord}_0 \mathcal{I}_{\tilde{\Delta}_{n-1}}$; but $\text{ord}_0 \mathcal{I}_{\tilde{\Delta}_{n-1}}$ is the minimal degree of generators of $\mathcal{I}_{\tilde{\Delta}_{n-1}}$. But \mathcal{I}_{Δ_n} and $\mathcal{I}_{\tilde{\Delta}_{n-1}}$ have the same generators, hence $\text{ord}_{\Delta_{1, \dots, n}} \mathcal{I}_{\Delta_n} = d_n$. Since the order of a coherent ideal along a subvariety is an holomorphic invariant, we can say in general that, for a smooth algebraic surface X ,

$$\text{ord}_{\Delta_{1, \dots, n}} \mathcal{I}_{\Delta_n} = d_n.$$

Remark 2.9. Consider $X = \mathbb{C}^2$. Note that, if $\{(p_i, q_i), i = 1, \dots, n-1\}$ is a set of $n-1$ distinct points in $\mathbb{N} \times \mathbb{N}$ not containing the origin, the polynomial $\Delta(\bar{p}, \bar{q}, \bar{z}, \bar{w})$ belongs to $\mathcal{I}_{\tilde{\Delta}_{n-1}}$.

2.2 F -pure thresholds

For computational convenience we consider the characteristic p analogue of the log-canonical threshold [TW04, MTW05]. Let k be a perfect field of characteristic p ; let R be a finitely generated regular k -algebra and $\mathfrak{a} \subseteq R$ a nonzero ideal; consider $M = \operatorname{Spec} R$ and let $x \in V(\mathfrak{a})$ be a closed point corresponding to a maximal ideal \mathfrak{m}_x . For $e \in \mathbb{N}^*$, define

$$\nu_{\mathfrak{a}}(e) := \max \left\{ i \in \mathbb{N} \mid \mathfrak{a}^i \not\subseteq \mathfrak{m}_x^{[p^e]} \right\}$$

where $\mathfrak{m}_x^{[p^e]}$ is the ideal generated by p^e -powers of generators of \mathfrak{m}_x . The inequality $\nu_{\mathfrak{a}}(e+1) \geq p\nu_{\mathfrak{a}}(e)$ implies that the sequences $\nu_{\mathfrak{a}}(e)/p^e$ and $\nu_{\mathfrak{a}}(e)/(p^e - 1)$ are nondecreasing [MTW05, Lemma 1.1]. The F -pure threshold of the ideal \mathfrak{a} at the point x is defined as

$$\operatorname{fpt}_x(M, \mathfrak{a}) := \lim_{e \rightarrow +\infty} \frac{\nu_{\mathfrak{a}}(e)}{p^e} = \lim_{e \rightarrow +\infty} \frac{\nu_{\mathfrak{a}}(e)}{(p^e - 1)} = \sup_{e \in \mathbb{N}^*} \frac{\nu_{\mathfrak{a}}(e)}{p^e} = \sup_{e \in \mathbb{N}^*} \frac{\nu_{\mathfrak{a}}(e)}{(p^e - 1)}. \quad (2.3)$$

Suppose now that \mathfrak{a} is principal: we write simply $\nu_f(e)$ instead of $\nu_{(f)}(e)$ and $\operatorname{fpt}_x(M, f)$ instead of $\operatorname{fpt}_x(M, (f))$. In this case the sequence $\nu_{\mathfrak{a}}(e)/p^e$ is bounded above by 1. Hence, for any $e \in \mathbb{N}^*$ we have the inequalities

$$\frac{\nu_f(e)}{(p^e - 1)} \leq \operatorname{fpt}_x(M, f) \leq 1. \quad (2.4)$$

Suppose now that M is the affine space $\mathbb{A}_{\mathbb{Z}}^n$ over \mathbb{Z} and \mathfrak{a} is a nonzero ideal of $R := \mathbb{Z}[x_1, \dots, x_n]$. For any prime p consider the mod p reduction $M_p := \operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{F}_p)$ and $\mathfrak{a}_p = \mathfrak{a} \cdot \mathbb{F}_p[x_1, \dots, x_n]$. On the other hand, if \mathbb{K} is an arbitrary field extension of \mathbb{Q} we can consider the extensions $\mathfrak{a}_{\mathbb{K}}$ inside $\mathbb{K}[x_1, \dots, x_n]$, respectively and $M_{\mathbb{K}} := \operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{K})$. For varieties defined over arbitrary perfect fields, Zhu recently proved an interpretation of the log-canonical threshold in terms of dimensions of jet-schemes [Zhu13, Theorem B]; this result yields, as a consequence, the inequality $\operatorname{fpt}_x(M_p, \mathfrak{a}_p) \leq \operatorname{lct}_x(M_{\mathbb{Q}}, \mathfrak{a}_{\mathbb{Q}})$ for every prime p and for every closed point $x \in V(\mathfrak{a})$ [Zhu13, Corollary 4.2]. Since the dimension of a scheme does not change upon extension of the field of definition [Gro65, Corollaire 4.1.4], we have, for every prime p and any closed point $x \in V(\mathfrak{a})$

$$\operatorname{fpt}_x(M_p, \mathfrak{a}_p) \leq \operatorname{lct}_x(M_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}). \quad (2.5)$$

2.3 Singularities of the isospectral Hilbert scheme

We begin by establishing the following upper bound for the log-canonical threshold of the pair $(X^n, \mathcal{I}_{\Delta_n})$.

Proposition 2.10. *The log-canonical threshold of $(X^n, \mathcal{I}_{\Delta_n})$ is bounded above by $(2n - 2)/d_n$:*

$$\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq \frac{2n - 2}{d_n}.$$

Proof. By remark 2.5 it is sufficient to prove the inequality when $X = \mathbb{C}^2$. By remark 2.8, for $c \in \mathbb{Q}$, $c > 0$, the order of $c \cdot \mathcal{I}_{\Delta_n}$ along the small diagonal $\Delta_{1, \dots, n}$ is cd_n ; as soon as $cd_n \geq \operatorname{codim}_X \Delta_{1, \dots, n} + 1 - 1 = 2n - 2$, that is, if $c \geq (2n - 2)/d_n$, by [Laz04, Example 9.3.7] we have that $\mathcal{J}(X, c \cdot \mathcal{I}_{\Delta_n}) \subseteq \mathcal{I}_{\Delta_{1, \dots, n}}$. By definition of log-canonical threshold $\operatorname{lct}_0(X^n, \mathcal{I}_{\Delta_n})$ as infimum, we get the desired inequality $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq \operatorname{lct}_0(X^n, \mathcal{I}_{\Delta_n}) \leq (2n - 2)/d_n$. \square

Remark 2.11. Consider the symmetric variety $S^n X$, where X is a smooth complex algebraic surface; we will indicate with $\pi : X^n \longrightarrow S^n X$ the quotient projection. It is well known that $S^n X$ admits a stratification in strata $S_{\lambda}^n X$, where λ is a partition of n . The stratum $S_{\lambda}^n X$ is the locally closed subset of 0-cycles of the form $\sum_{i=1}^{l(\lambda)} \lambda_i x_i$, where $l(\lambda)$ is the length of the partition λ and x_i are $l(\lambda)$ distinct points in X . By means of this stratification of $S^n X$ we can define a stratification of X^n setting the stratum X_{λ}^n as the locally closed subset $\pi^{-1}(S_{\lambda}^n X)$. It is clear that if $x \in X_{\lambda}^n$ then a sufficiently small open set V_1 of x in X^n in the standard topology is biholomorphic to a sufficiently small open set V_2 of the origin in $(\mathbb{C}^2)^n$ of the form $V_2 = U_1^{\lambda_1} \times \dots \times U_{l(\lambda)}^{\lambda_{l(\lambda)}}$, where U_i are adequate small open sets of the origin in \mathbb{C}^2 , such that, via the biholomorphic map, the ideal \mathcal{I}_{Δ_n} over V_1 is sent to $\mathcal{I}_{\Delta_{\lambda_1}} \boxtimes \dots \boxtimes \mathcal{I}_{\Delta_{\lambda_{l(\lambda)}}}$ over V_2 . Therefore, if $x \in X_{\lambda}^n$, we have, by proposition 2.10 and by [Laz04, Proposition 9.5.22] that

$$\operatorname{lct}_x(X^n, \mathcal{I}_{\Delta_n}) = \min \left\{ \operatorname{lct}_0((\mathbb{C}^2)^{\lambda_i}, \mathcal{I}_{\Delta_{\lambda_i}}) \mid i = 1, \dots, l(\lambda) \right\} \leq \frac{2\lambda_1 - 2}{d_{\lambda_1}}. \quad (2.6)$$

We now make the following conjecture

Conjecture 1. *Let X be a smooth algebraic surface. If a point x of X^n lies in the stratum X_λ^n , where λ is a partition of n , then $\text{lct}_x(X^n, \mathcal{I}_{\Delta_n}) = (2\lambda_1 - 2)/d_{\lambda_1}$. Therefore*

$$\text{lct}(X^n, \mathcal{I}_{\Delta_n}) = \frac{2n - 2}{d_n}.$$

This conjecture would immediately imply the following fact about the singularities of the isospectral Hilbert scheme B^n .

Conjecture 2. *The singularities of the isospectral Hilbert scheme B^n are canonical if and only if $n \leq 7$, log-canonical if $n \leq 8$, not log-canonical if $n \geq 9$.*

We are able to partially prove conjecture 2.

Theorem 2.12. *The singularities of the isospectral Hilbert scheme B^n are canonical if $n \leq 5$, log-canonical if $n \leq 7$. For $n \geq 9$ they are not log-canonical.*

Proof. By corollary 2.3 and by remark 2.4 the singularities of the isospectral Hilbert scheme B^n are log-canonical if and only if $\text{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1$ and canonical if and only if $\text{lct}(X^n, \mathcal{I}_{\Delta_n}) > 1$. For $n \geq 9$, by proposition 2.10, $\text{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq (2n - 2)/d_n \leq 16/17$. Hence they can't be log-canonical.

Let's now prove the first statement. Using corollary 2.3 and remark 2.4 it is sufficient to prove that the singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$ are canonical for $n \leq 5$ and that $\text{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1$ for $n = 6, 7$. By remark 2.5 it is sufficient to prove these facts for $X = \mathbb{C}^2$. By (2.2), it is then sufficient to prove that the pair $(\mathbb{C}^{2n-2}, \mathcal{I}_{\tilde{D}_{n-1}})$ has canonical singularities for $n \leq 5$ and is log-canonical for $n = 6, 7$.

To prove that the pair $(\mathbb{C}^{2n-2}, \mathcal{I}_{\tilde{D}_{n-1}})$ is canonical for $n \leq 4$ we will use Kollar-Bertini theorem [Kol97, Theorems 4.5, 4.5.1], [Laz04, Example 9.3.50]: in other words we will find a $g \in \mathcal{I}_{\tilde{D}_{n-1}}$ such that $\text{div } g$ has rational (or canonical) singularities; then Kollar-Bertini theorem implies that the pair $(\mathbb{C}^{2n-2}, \mathcal{I}_{\tilde{D}_{n-1}})$ is canonical. For $n = 3$ such a g can be chosen as the generator of minimal degree of $\mathcal{I}_{\tilde{D}_2}$, that is, $g = z_1 w_2 - z_2 w_1$: it defines an affine quadric cone of in \mathbb{C}^4 projecting a smooth quadric in \mathbb{P}^3 from the origin of \mathbb{C}^4 . Hence, by [Bur74, Example 1.2], it has rational singularities. For $n = 4$ we can use the generator of minimal degree of $\mathcal{I}_{\tilde{D}_3}$ given by the polynomial $g = \Delta((1, 0, 1), (0, 1, 1), \bar{z}, \bar{w})$. One can show that g has rational singularities using *Macaulay2* [GS] and, in particular, the command `hasRationalSing` of the package `D-modules`.

For $n \geq 5$ it is computationally more efficient to use characteristic p methods. Let now $n = 5$. By the equality in (2.6) and by what we just proved, we know that for any point x in a strata X_λ^5 , with $\lambda \neq (5)$, we have $\text{lct}_x(X^5, \mathcal{I}_{\Delta_5}) \geq \text{lct}(X^4, \mathcal{I}_{\Delta_4}) > 1$. It is then sufficient to prove that, for a point $x \in \Delta_{1, \dots, 5}$, $\text{lct}_x(\mathbb{C}^{10}, \mathcal{I}_{\Delta_5}) > 1$. Because of the isomorphism (2.2) it is sufficient to prove that $\text{lct}_0(\mathbb{C}^8, \mathcal{I}_{\tilde{D}_4}) > 1$. By (2.5) it is sufficient to prove, for some prime p , that $\text{fpt}_0((\mathbb{F}_p^2)^4, (\mathcal{I}_{\tilde{D}_4})_p) > 1$. Consider the polynomials $g = \Delta((1, 0, 2, 1), (0, 1, 0, 2), \bar{z}, \bar{w})$ and $h = \Delta((1, 0, 2, 0), (0, 1, 0, 2), \bar{z}, \bar{w})$ in $\mathcal{I}_{\tilde{D}_4}$; we can check, using *Macaulay2* and passing modulo $p = 7$, that the class of $g^2 h^5$ is nonzero in $\mathbb{F}_7[z_1, \dots, z_4, w_1, \dots, w_4]/\mathfrak{m}_0^{[7]}$, thus proving that $\nu_{\mathfrak{a}}(1) \geq 7$, where $\mathfrak{a} = (\mathcal{I}_{\tilde{D}_4})_7$, and hence that $\text{fpt}_0((\mathbb{F}_7^2)^4, (\mathcal{I}_{\tilde{D}_4})_7) \geq 7/6 > 1$, by (2.3). Therefore the pair $(X^5, \mathcal{I}_{\Delta_5})$ has canonical singularities.

Let now $n = 6, 7$. By the equality in (2.6) and by what we just proved, we already know that for any point x in a stratum X_λ^n , with $\lambda \neq (6)$ — in the case $n = 6$ — or $\lambda \neq (7)$ and $\lambda \neq (6, 1)$ — in the case $n = 7$ — we have $\text{lct}_x(X^n, \mathcal{I}_{\Delta_n}) \geq \text{lct}(X^5, \mathcal{I}_{\Delta_5}) > 1$. For $n = 6$ it is then sufficient to prove that $\text{lct}_x(\mathbb{C}^{12}, \mathcal{I}_{\Delta_6}) \geq 1$ when $x \in \Delta_{1, \dots, 6}$; by the isomorphism (2.2), it is sufficient to prove that $\text{lct}_0(\mathbb{C}^{10}, \mathcal{I}_{\tilde{D}_5}) \geq 1$; once we prove it, it is sufficient to prove that $\text{lct}_x(\mathbb{C}^{14}, \mathcal{I}_{\Delta_7}) > 1$ for $x \in \Delta_{1, \dots, 7}$, or equivalently, after (2.2), that $\text{lct}_0(\mathbb{C}^{12}, \mathcal{I}_{\tilde{D}_6}) \geq 1$. By (2.5) it is sufficient to prove, for some prime p , that $\text{fpt}_0((\mathbb{F}_p^2)^{n-1}, (\mathcal{I}_{\tilde{D}_{n-1}})_p) \geq 1$ for $n = 6, 7$. By the first of the inequalities (2.4) it is then sufficient to find a polynomial $g \in \mathcal{I}_{\tilde{D}_{n-1}}$, with integral coefficients, such that, for some prime p , $\nu_{g_p}(1) = p - 1$ at the origin: here, for a polynomial g with integral coefficients, we denote with g_p its mod p reduction in $(\mathcal{I}_{\tilde{D}_{n-1}})_p$. Consider the polynomials with integral coefficients $g = \Delta((1, 0, 2, 1, 0), (0, 1, 0, 1, 2), \bar{z}, \bar{w})$, for $n = 6$, and $h = \Delta((1, 0, 2, 1, 0, 2), (0, 1, 0, 1, 2, 1), \bar{z}, \bar{w})$, for $n = 7$. Then, passing modulo $p = 7$, we checked, using *Macaulay2*, that the classes of g^6 in $\mathbb{F}_7[z_1, \dots, z_5, w_1, \dots, w_5]/\mathfrak{m}_0^{[7]}$ and h^6 in $\mathbb{F}_7[z_1, \dots, z_6, w_1, \dots, w_6]/\mathfrak{m}_0^{[7]}$ are both non zero. This proves that, choosing the prime $p = 7$, $\nu_{g_7}(1) = 6 = \nu_{h_7}(1)$ and we can conclude. \square

2.4 Relation with the geometry of the Hilbert scheme of points

The geometry of the pair $(X^n, \mathcal{I}_{\Delta_n})$ is not only directly related to the geometry of the isospectral Hilbert scheme B^n , but also to the geometry of the Hilbert scheme of n points $X^{[n]}$ over the surface X . Consider the boundary $\partial X^{[n]}$ of $X^{[n]}$. Song proved in [Son14, Proposition 4.3.5] that

$$\mathrm{lct}(X^{[n]}, \mathcal{I}_{\partial X^{[n]}}) = \mathrm{lct}(S^n X, \mathcal{I}_{\Delta_n}^{\mathfrak{S}_n}) = \frac{1}{2} \mathrm{lct}(X^n, \mathcal{I}_{\Delta_n}).$$

Hence proposition 2.10 implies immediately the

Corollary 2.13. *The log-canonical threshold of the pair $(X^{[n]}, \mathcal{I}_{\partial X^{[n]}})$ is bounded above by $(n-1)/d_n$.*

Moreover, conjecture 1 would imply

Conjecture 3. *The log-canonical threshold of the pair $(X^{[n]}, \mathcal{I}_{\partial X^{[n]}})$ is precisely given by $(n-1)/d_n$.*

3 Two resolutions of B^3

The aim of this subsection is to provide two explicit resolutions of singularities of B^3 ; the first will be *crepant*, the second will be \mathfrak{S}_3 -equivariant. We begin with some remarks and technical lemmas.

Remark 3.1. Let M be a smooth algebraic variety and let F be a coherent sheaf over M . We recall that an integral subscheme V of M is called a *prime cycle associated to F* if there exists an invertible coherent \mathcal{O}_V -module L and an embedding $L \hookrightarrow F$ of coherent \mathcal{O}_M -modules.

Remark 3.2. Let M be a smooth algebraic variety and Y a smooth subvariety. Let $Z \subseteq M$ be a closed subscheme, defined by the ideal sheaf \mathcal{I}_Z . Let $r = \mathrm{ord}_Y \mathcal{I}_Z$ the order of Z along Y . Consider the blow-up $f : \mathrm{Bl}_Y M \rightarrow M$ of Y in M and denote with E its exceptional divisor. The *weak transform* \tilde{Z} of Z in $\mathrm{Bl}_Y M$ is defined by the residual ideal $\mathcal{I}_{\tilde{Z}} := (\mathcal{I}_{f^{-1}(Z)} : \mathcal{I}_E^r)$. The ideal of the total transform $f^{-1}(Z)$ is then given by the product

$$\mathcal{I}_{f^{-1}(Z)} = \mathcal{I}_E^r \cdot \mathcal{I}_{\tilde{Z}}.$$

It is well known that the weak transform does not necessarily coincide with the strict transform \hat{Z} ; in general one just has that $\mathcal{I}_{\tilde{Z}} \subseteq \mathcal{I}_{\hat{Z}}$, and that the two ideals coincide outside the exceptional divisor. Indeed the weak transform \tilde{Z} could contain embedded components over the exceptional divisor, while the strict transform doesn't. This is, in any case, the only possible difference between \tilde{Z} and \hat{Z} , as the next criterion proves.

Proposition 3.3. *Let M be a smooth algebraic variety and Y a smooth subvariety. Let $Z \subseteq M$ be a closed subscheme. Consider the blow-up map $f : \mathrm{Bl}_Y M \rightarrow M$ and let E be the exceptional divisor. Then the weak transform \tilde{Z} of Z coincides with the strict transform \hat{Z} if and only if E does not contain any prime cycle associated to \tilde{Z} . In this case, for any positive integer l , the subschemes lE and \hat{Z} are transverse.*

Proof. The necessity of the condition is clear. We just have to prove the sufficiency. Recall that the strict transform \hat{Z} can be identified with the blow-up $\mathrm{Bl}_{Y \cap Z} Z$: this is a consequence, for example, of [EH00, Proposition IV-21]. Indicate with λ the canonical section of $\mathcal{O}_{\mathrm{Bl}_Y M}(E)$. We have that E does not contain prime cycles associated to \tilde{Z} if and only if the morphism $\lambda : \mathcal{O}_{\tilde{Z}}(-E) \rightarrow \mathcal{O}_{\tilde{Z}}$ is injective. In this case the ideal $\mathcal{I}_{\tilde{Z} \cap E / \tilde{Z}}$ of $\tilde{Z} \cap E$ in \tilde{Z} is an invertible ideal of $\mathcal{O}_{\tilde{Z}}$. Hence the map $f|_{\tilde{Z}} : \tilde{Z} \rightarrow Z$ factors via the blow-up $\mathrm{Bl}_{Y \cap Z} Z$, that is, via the strict transform \hat{Z} . Hence we have the injection of schemes $\tilde{Z} \hookrightarrow \hat{Z}$. But it is always true that $\hat{Z} \subseteq \tilde{Z}$. Hence the weak transform coincides with the strict one. In this case, for any fixed positive integer l , the morphism $\lambda^l : \mathcal{O}_{\tilde{Z}}(-lE) \rightarrow \mathcal{O}_{\tilde{Z}}$ is injective. Since $R^\bullet := 0 \rightarrow \mathcal{O}_{\mathrm{Bl}_Y M}(-lE) \rightarrow \mathcal{O}_{\mathrm{Bl}_Y M}$ is a locally free resolution of \mathcal{O}_{lE} , we can compute $\mathrm{Tor}_j(\mathcal{O}_{lE}, \mathcal{O}_{\tilde{Z}})$ as of the $(-j)$ -cohomology of the complex $R^\bullet \otimes \mathcal{O}_{\tilde{Z}}$, which is $0 \rightarrow \mathcal{O}_{\tilde{Z}}(-lE) \xrightarrow{\lambda^l} \mathcal{O}_{\tilde{Z}} \rightarrow 0$. Hence $\mathrm{Tor}_j(\mathcal{O}_{lE}, \mathcal{O}_{\tilde{Z}}) = 0$ for $j > 0$. \square

Remark 3.4. Let M be a smooth algebraic variety, and Y a smooth subvariety. Consider the blow-up map $f : \mathrm{Bl}_Y M \rightarrow M$. Let H be an hypersurface in M . Then its weak and strict transform in $\mathrm{Bl}_Y M$ coincide.

Proof. Let E be the exceptional divisor. The weak transform \tilde{H} is a divisor whose associated prime cycles are the irreducible components of \tilde{H} . Since, by definition of \tilde{H} , one has that $E \not\subset \tilde{H}$, then $\text{codim}_{\text{Bl}_Y M} E \cap \tilde{H} = 2$ and hence the local equations of E and \tilde{H} define a regular sequence; hence E does not contain any prime cycles relative to \tilde{H} . Hence $\tilde{H} = \hat{H}$. \square

Lemma 3.5. *Let M be a smooth algebraic variety and let Y, W, Z three subschemes of M , such that Y is closed, W is integral and that $Y \not\subseteq W$. Let \widehat{W}, \widehat{Z} be the strict transforms of W and Z inside $\text{Bl}_Y M$. Then $\text{ord}_W \mathcal{I}_Z = \text{ord}_{\widehat{W}} \mathcal{I}_{\widehat{Z}}$.*

Proof. Note that if S, T are two subschemes of a smooth algebraic variety V , with T integral, then $\text{ord}_T \mathcal{I}_S$ can be characterized as $\text{ord}_T \mathcal{I}_S = \max\{n \in \mathbb{N} \mid \mathcal{I}_{S,T} \subseteq \mathfrak{m}_T^n\}$ where \mathfrak{m}_T is the maximal ideal of the local ring $\mathcal{O}_{V,T}$ — that is, the ring of regular functions g defined on some open set U intersecting T [Har77, Exercise 3.13] — and where $\mathcal{I}_{S,T}$ is the ideal of functions g in $\mathcal{O}_{V,T}$ vanishing over $S \cap U$, if U is the open set of definition of g . Now the blow-up map $f : \text{Bl}_Y M \longrightarrow M$ induces an isomorphism of local rings $f_W^* : \mathcal{O}_{M,W} \longrightarrow \mathcal{O}_{\text{Bl}_Y M, \widehat{W}}$ under which $\mathcal{I}_{Z,W}$ is sent onto $\mathcal{I}_{\widehat{Z}, \widehat{W}}$, hence the statement. \square

Lemma 3.6. *Let M be a smooth algebraic variety of dimension at least 3; let H be a smooth hypersurface in M and W_1, W_2 two smooth subvarieties of M contained in H and transverse inside H . Consider now the composition f of blow-ups*

$$f : B := \text{Bl}_{\widehat{W}_2} \text{Bl}_{W_1} M \xrightarrow{f_2} \text{Bl}_{W_1} M \xrightarrow{f_1} M,$$

where \widehat{W}_2 is the strict transform of W_2 inside $\text{Bl}_{W_1} M$. Denote with E_{W_1} the exceptional divisor of $\text{Bl}_{W_1} M$ and with $E_{\widehat{W}_2}$ that of $\text{Bl}_{\widehat{W}_2} \text{Bl}_{W_1} M$. Then f is an isomorphism outside $f^{-1}(W_1 \cup W_2)$; moreover

$$f^{-1}(\mathcal{I}_{W_1 \cup W_2}) = \mathcal{I}_{\widehat{E}_{W_1}} \cdot \mathcal{I}_{E_{\widehat{W}_2}} = \mathcal{O}_B(-\widehat{E}_{W_1} - E_{\widehat{W}_2}).$$

Finally the relative canonical bundle $K_{B/M}$ is isomorphic to $\mathcal{O}_B(\widehat{E}_{W_1} + E_{\widehat{W}_2})$.

Proof. In the particular case in which $M = \mathbb{C}^3$; $\mathcal{I}_H = (x)$; $\mathcal{I}_{W_1} = (x, y)$; $\mathcal{I}_{W_2} = (x, z)$ and hence $\mathcal{I}_{W_1 \cup W_2} = (x, yz)$, the statement can be proved by an explicit computation in coordinates, which we leave to the reader.

Let's now pass to the general case. Consider a point p in the intersection $W_1 \cap W_2$. Over an adequate open neighbourhood U of p in the standard complex topology, we can find local holomorphic coordinates x, y, z such that H is defined (over U) by the zeros of x , and W_1 and W_2 by the ideals (x, y) and (x, z) , respectively. Alternatively, one can find an adequate affine neighbourhood U of p and regular function x, y, z over U such that the differentials dx, dy, dz are independent in $\mathfrak{m}_q/\mathfrak{m}_q^2$ for all $q \in U$ and such that H, W_1, W_2 are defined by ideals of the regular functions $(x), (x, y)$ and (x, z) as in the holomorphic case. Hence the general situation can be obtained locally from the particular one above by a smooth base change: the statement follows. \square

Lemma 3.7. *Let M be a smooth algebraic variety, H a smooth hypersurface of M , and W and Q two codimension 2 smooth subvarieties of M such that $Q \subseteq H$, $W \cap H \subseteq Q$ and $W \cap H$ is a smooth codimension 3 subvariety of M . Consider the blow-up $f : \text{Bl}_W M \longrightarrow M$ of W in M , with exceptional divisor E_W . Then*

$$f^{-1}(\mathcal{I}_W \cap \mathcal{I}_Q) = \mathcal{I}_{E_W} \cdot \mathcal{I}_{\widehat{Q}} = \mathcal{I}_{E_W} \cap \mathcal{I}_{\widehat{Q}}$$

where \widehat{Q} denote the strict transform of Q in $\text{Bl}_W M$.

Proof. The statement is local in nature, over the base M : hence, by placing ourselves on a small open neighbourhood of a point $p \in W \cap H$ in the complex topology, equipped with some holomorphic coordinates $(x, y, z, w_1, \dots, w_r)$, we can suppose that the ideals of H, W and Q are given locally by $\mathcal{I}_H = (z)$, $\mathcal{I}_W = (x, y)$, $\mathcal{I}_Q = (x, z)$. Then $\mathcal{I}_W \cap \mathcal{I}_Q = (x, yz)$; the proof of the statement is now achieved through an easy computation in coordinates. \square

3.1 A crepant resolution of B^3 .

Conjecture 1 states that the log-canonical threshold of the pair $(X^3, \mathcal{I}_{\Delta_3})$ is 2. This fact suggests that B^3 might admit a crepant resolution. This is indeed the case, as we will prove in this subsection.

Remark 3.8. Let X be a smooth algebraic surface. If Y is any smooth variety admitting a projective birational morphism $f : Y \longrightarrow X^n$ over X^n such that $f^{-1}(\mathcal{I}_{\Delta_n})$ is an invertible ideal sheaf of \mathcal{O}_Y , then, by the universal property of the blow-up, the map f factors via the isospectral Hilbert scheme B^n as

$$\begin{array}{ccc} Y & & \\ \downarrow h & \searrow f & \\ B^n & \xrightarrow{p} & X^n \end{array}$$

providing a resolution h of B^n such that

$$K_Y - h^*K_{B^n} = K_Y - h^*(p^*K_{X^n} + \mathcal{O}_{B^n}(E)) = K_Y - f^*K_{X^n} + h^{-1}\mathcal{I}_E = K_Y - f^*K_{X^n} + f^{-1}(\mathcal{I}_{\Delta_n}).$$

Remark 3.9. By the previous remark, in order to find a crepant resolution of B^n , it is sufficient to build a smooth variety Y and a projective birational map $f : Y \longrightarrow X^n$ such that $f^{-1}(\mathcal{I}_{\Delta_n})$ is an invertible ideal isomorphic to the relative anticanonical $-K_{Y/X^n} = f^*K_{X^n} - K_Y$.

Remark 3.10. The questions posed in the previous two remarks are local over the base and analytical in nature. Hence, to find a resolution of B^n in general, it is sufficient to find a smooth variety Y and a birational map as in the remark 3.8 for $X = \mathbb{C}^2$. Moreover, since in the identification (2.2), the ideal sheaf \mathcal{I}_{Δ_n} corresponds to $\mathcal{I}_{\tilde{D}_{n-1}} \boxtimes \mathcal{O}_{\mathbb{C}^2}$, by flat base change it is sufficient to find a smooth variety Y and a projective birational morphism $f : Y \longrightarrow (\mathbb{C}^2)^{n-1}$ such that $f^{-1}(\mathcal{I}_{\tilde{D}_{n-1}})$ is an invertible ideal. The resolution thus built will be crepant if and only if $f^{-1}(\mathcal{I}_{\tilde{D}_{n-1}})$ is isomorphic to the anticanonical $-K_Y$.

For brevity's sake, in what follows, we will indicate the affine space $(\mathbb{C}^2)^2$ with V , the subscheme \tilde{D}_2 with W . Fix coordinates (x, y, z, w) over V . The irreducible components of the subscheme W are linear subspaces W_1, W_2, W_3 , defined by the ideals $I_1 = (x, y)$, $I_2 = (z, w)$, $I_3 = (x - z, y - w)$. The ideal \mathcal{I}_W is then given by $\langle q, I_1 I_2 I_3 \rangle$, where q is the quadric $q = xw - yz$.

Proposition 3.11. *The projective birational morphism $f : Y \longrightarrow V$, defined as the composition of smooth blow-ups*

$$Y = Y_3 \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} V$$

where $Y_1 = \text{Bl}_{W_1} V$, $Y_2 = \text{Bl}_{\widehat{W}_2} Y_1$, $Y_3 = \text{Bl}_{\widehat{W}_3} Y_2$, where $\widehat{W}_2, \widehat{W}_3$ are the strict transforms of W_2, W_3 in Y_1, Y_2 , respectively, is an isomorphism outside the locus $f^{-1}(W)$. Moreover, the ideal sheaf $f^{-1}(\mathcal{I}_W)$ is invertible and isomorphic to $-K_Y$.

Proof. As generators of the ideal \mathcal{I}_W we can choose the polynomials $q, xz(x - z), xw(y - w), yw(x - z), yw(y - w)$. Consider the first blow-up $Y_1 = \text{Bl}_{W_1} V \simeq \text{Bl}_0(\mathbb{C}^2) \times \mathbb{C}^2$ and denote with E_1 the exceptional divisor. We can write globally

$$x = \lambda u, \quad y = \lambda v$$

where λ is the canonical section of $\mathcal{O}_{Y_1}(E_1)$ and u, v are homogeneous coordinates, thought as a basis in $H^0(\mathcal{O}_{Y_1}(-E_1))$. By definition of weak transform we have $\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_1} \cdot \mathcal{I}_{\widehat{W}}$. The weak transform \widehat{W} is given by the equations

$$\begin{aligned} uw - vz &= 0 \\ uz(\lambda u - z) &= 0 \\ uw(\lambda v - w) &= 0 \\ vw(\lambda u - z) &= 0 \\ vw(\lambda v - w) &= 0 \end{aligned}$$

We prove now that the weak transform \widetilde{W} coincides with the strict transform \widehat{W} . By proposition 3.3 and its proof we just have to show that the morphism $\lambda : \mathcal{O}_{\widetilde{W}}(-E_1) \longrightarrow \mathcal{O}_{\widetilde{W}}$ is injective. Now, \widetilde{W} is contained in the hypersurface H of Y_1 defined by the equation $uw - vz = 0$. Over H we can globally write $z = \mu u$, $w = \mu v$, where μ can be seen as a section in $H^0(\mathcal{O}_H(E_1))$. Then \widetilde{W} is given, inside H , by the equations

$$\begin{aligned} u^3 \mu(\lambda - \mu) &= 0 \\ uv^2 \mu(\lambda - \mu) &= 0 \\ uv^2 \mu(\lambda - \mu) &= 0 \\ v^3 \mu(\lambda - \mu) &= 0 \end{aligned}$$

Since u and v do not vanish at the same time, the weak transform is given by the equation $\mu(\lambda - \mu) = 0$ inside the hypersurface H , with respect to the coordinates $([u, v], \lambda, \mu)$. Hence λ is not zero divisor in \widetilde{W} and $\widetilde{W} = \widehat{W}$. Hence

$$\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_1} \cdot \mathcal{I}_{\widehat{W}}.$$

Now \widehat{W} is clearly the union, inside H , of the two smooth surfaces \widehat{W}_2 and \widehat{W}_3 intersecting transversally along a smooth curve inside the exceptional divisor E_1 . Consider now the blow-ups $f_2 : \text{Bl}_{\widehat{W}_2} Y_1 \longrightarrow Y_1$, with exceptional divisor E_2 , and $f_3 : \text{Bl}_{\widehat{W}_3} Y_2 \longrightarrow Y_2$, with exceptional divisor E_3 ; denote with $\widehat{\widehat{E}}_1$ and $\widehat{\widehat{E}}_2$ the strict transforms of E_1 and E_2 in Y_3 , respectively. Let now $g := f_2 \circ f_3$ and let $f := f_1 \circ g$. Then by lemma 3.6 we have

$$f^{-1}(\mathcal{I}_W) = g^{-1}(\mathcal{I}_{f_1^{-1}(W)}) = g^{-1}(\mathcal{I}_{E_1}) \cdot g^{-1}(\mathcal{I}_{\widehat{W}}) = \mathcal{I}_{\widehat{\widehat{E}}_1} \cdot \mathcal{I}_{\widehat{\widehat{E}}_2} \cdot \mathcal{I}_{E_3},$$

where we used that $\widetilde{E}_1 = \widehat{E}_1$ and $\widetilde{\widehat{E}}_1 = \widehat{\widehat{E}}_1$ by remark 3.4. Hence $f^{-1}(\mathcal{I}_W)$ is invertible and isomorphic to $\mathcal{O}_Y(-\widehat{\widehat{E}}_1 - \widehat{\widehat{E}}_2 - E_3)$; it is now easy to show that the latter coincides with the anticanonical divisor $-K_Y$. \square

As an immediate consequence of remarks 3.8, 3.9 and 3.10 we deduce the

Corollary 3.12. *The map $f : Y \longrightarrow V$ factors through a crepant resolution $h : Y \longrightarrow \text{Bl}_W V$. Consequently the map $h \times \text{id} : Y \times \mathbb{C}^2 \longrightarrow \text{Bl}_W V \times \mathbb{C}^2 \simeq B^3$ identifies to a crepant resolution of B^3 .*

Let now X be an arbitrary smooth algebraic surface and let $\Delta_{I_1}, \Delta_{I_2}, \Delta_{I_3}$ be the pairwise diagonals Δ_I , $|I| = 2$, taken in whatever order. We have the following

Theorem 3.13. *The composition of blow-ups $s := s_1 \circ s_2 \circ s_3$*

$$Y := \text{Bl}_{\widehat{\Delta}_{I_3}} Y_2 \xrightarrow{s_3} Y_2 := \text{Bl}_{\widehat{\Delta}_{I_2}} Y_1 \xrightarrow{s_2} Y_1 := \text{Bl}_{\Delta_{I_1}} X^3 \xrightarrow{s_1} X^3$$

where $\widehat{\Delta}_{I_2}$ and $\widehat{\Delta}_{I_3}$ are the strict transforms of Δ_{I_2} and Δ_{I_3} in Y_1 and Y_2 , respectively, is a log-resolution of the pair $(X^3, \mathcal{I}_{\Delta_3})$ such that $s^{-1}(\mathcal{I}_{\Delta_3})$ is an invertible ideal isomorphic to the relative anticanonical $-K_{Y/X^3}$. Hence s factors through a crepant resolution $g : Y \longrightarrow B^3$ of the isospectral Hilbert scheme B^3 .

Proof. Locally over X^3 , the map s coincides precisely with $\varphi^{-1} \circ (f \times \text{id}_{\mathbb{C}^2})$, where f is the birational map built in theorem 3.11 and φ is the map (2.1). The theorem is then an immediate consequence of proposition 3.11 and remarks 3.8 and 3.9. \square

3.2 An \mathfrak{S}_3 -equivariant resolution of B^3

Consider the 4-dimensional vector space $V = (\mathbb{C}^2)^2$ with coordinates (x, y, z, w) and the subscheme $W = W_1 \cup W_2 \cup W_3$ introduced in subsection 3.1. Consider the blow-up $f_1 : Y_1 := \text{Bl}_0(V) \longrightarrow V$ of V at the origin and let E_0 be its exceptional divisor; since it can be identified with the total space of the Hopf line bundle over the projective space $\mathbb{P}(V)$, the variety Y_1 is equipped with a fibration $Y_1 \longrightarrow \mathbb{P}(V)$. Now, the polynomial $q = xw - yz$ defines a smooth quadric Q in $\mathbb{P}(V)$, which can be seen as a smooth subvariety of Y_1 inside E_0 , thanks to the embedding of $\mathbb{P}(V)$ into Y_1 given by the zero section of the Hopf bundle.

Proposition 3.14. *The birational morphism $f : Y \longrightarrow V$ defined as the composition of smooth blow-ups*

$$Y = Y_3 \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} V$$

where $Y_2 = \text{Bl}_{\widehat{W}}(Y_1)$, $Y_3 = \text{Bl}_{\widehat{Q}}(Y_2)$, where \widehat{W} and \widehat{Q} are the strict transforms of W and Q in Y_1 and Y_2 , respectively, is an isomorphism outside $f^{-1}(W)$. Moreover the ideal sheaf $f^{-1}(\mathcal{I}_W)$ is given by

$$f^{-1}(\mathcal{I}_W) = \mathcal{O}_Y(-2\widehat{\widehat{E}}_0 - \widehat{E}_{\widehat{W}} - 3E_{\widehat{Q}})$$

where $E_{\widehat{W}}$ and $E_{\widehat{Q}}$ are the exceptional divisors in Y_2 and Y_3 , respectively, and where $\widehat{\widehat{E}}_0$ and $\widehat{\widehat{E}}_0$ are the strict transforms of $E_{\widehat{W}}$ and E_0 in Y .

Proof. Since $\text{ord}_0 \mathcal{I}_W = 2$, we have

$$\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_0}^2 \cdot \mathcal{I}_{\widehat{W}}$$

where \widehat{W} is the weak transform of W in Y_1 . By a computation in coordinates, using the same generators for \mathcal{I}_W we used in the proof of theorem 3.11, one gets

$$\mathcal{I}_{\widehat{W}} = \mathcal{I}_Q \cap \mathcal{I}_{\widehat{W}},$$

that is, the weak transform \widehat{W} is the scheme-theoretic union of the quadric Q and the strict transform \widehat{W} of W in Y_1 , which is a smooth codimension 2 subvariety with three irreducible components \widehat{W}_i , $i = 1, \dots, 3$. Moreover $\widehat{W} \cap E_0$ is contained in Q and is precisely the union of three skew lines in $E_0 \simeq \mathbb{P}(V)$; hence $\widehat{W} \cap E_0$ is a smooth codimension 3 subvariety of Y_1 . Therefore the hypothesis of lemma 3.7 are satisfied; this means that, when blowing up the strict transform \widehat{W} in Y_1 one gets

$$f_2^{-1}(\mathcal{I}_Q \cap \mathcal{I}_{\widehat{W}}) = \mathcal{I}_{E_{\widehat{W}}} \cdot \mathcal{I}_{\widehat{Q}}.$$

Since $\text{ord}_{\widehat{W}} \widehat{E}_0 = 0$, we get

$$(f_1 \circ f_2)^{-1}(\mathcal{I}_W) = \mathcal{I}_{E_0}^2 \cdot \mathcal{I}_{E_{\widehat{W}}} \cdot \mathcal{I}_{\widehat{Q}}.$$

Remembering that $\text{ord}_{\widehat{Q}} \widehat{E}_0 = 1$, the last blow-up now yields the formula in the statement. \square

Corollary 3.15. *The map $f : Y \longrightarrow V$ factors through a resolution $h : Y \longrightarrow \text{Bl}_W V$. Consequently the map $h \times \text{id} : Y \times \mathbb{C}^2 \longrightarrow \text{Bl}_W V \times \mathbb{C}^2 \simeq B^3$ identifies to an \mathfrak{S}_3 -equivariant resolution of B^3 .*

Consider now the case of an arbitrary smooth algebraic surface X . Consider the blow-up $s_1 : Y_1 := \text{Bl}_{\Delta_{123}} X^3 \longrightarrow X^3$ of the small diagonal Δ_{123} in X^3 and let E_0 be its exceptional divisor. The situation is locally, over X^3 , analogous to the one just studied. Hence it is now clear that $s_1^{-1}(\mathcal{I}_{\Delta_3}) = \mathcal{I}_{E_0}^2 \cdot (\mathcal{I}_Q \cap \mathcal{I}_{\widehat{\Delta}_3})$, where $\widehat{\Delta}_3$ is the strict transform of Δ_3 in Y_1 and where Q is a quadric subbundle of $\mathbb{P}(N_{\Delta_{123}/X^3})$ over Δ_{123} and hence a smooth subvariety of Y_1 inside E_0 . We have the following theorem

Theorem 3.16. *The composition of smooth blow-ups $s := s_1 \circ s_2 \circ s_3$:*

$$Y := \text{Bl}_{\widehat{Q}} Y_2 \xrightarrow{s_3} Y_2 := \text{Bl}_{\widehat{\Delta}_3} Y_1 \xrightarrow{s_2} Y_1 \xrightarrow{s_1} X^3$$

where $\widehat{\Delta}_3$ and \widehat{Q} are the strict transforms of Δ_3 and Q in Y_1 and Y_2 , respectively, defines a \mathfrak{S}_3 -equivariant log-resolution of the pair $(X^3, \mathcal{I}_{\Delta_3})$ and hence factors through a \mathfrak{S}_3 -equivariant log-resolution $g : Y \longrightarrow B^3$ of the isospectral Hilbert scheme B^3 .

Proof. The map s is clearly \mathfrak{S}_3 -equivariant and, locally over X^3 , coincides with the map $\varphi^{-1} \circ (f \times \text{id}_{\mathbb{C}^2})$, where f is the map introduced in proposition 3.14 and where φ is the map (2.1). The content of the theorem is then a consequence of proposition 3.14, corollary 3.15 and remarks 3.8 and 3.9. \square

Remark 3.17. This resolution is not crepant, as one gets easily $K_{Y/X^3} + s^{-1}(\mathcal{I}_{\Delta_3}) = \mathcal{O}(\widehat{\widehat{E}}_0 + E_{\widehat{Q}})$, where $E_{\widehat{Q}}$ is the exceptional divisor in Y_3 and where $\widehat{\widehat{E}}_0$ is the strict transform of E_0 in Y .

Remark 3.18. The step Y_2 coincides with the Fulton-MacPherson compactification $X[3]$ of $X^3 \setminus \Delta_3$ (see [FM94]).

Remark 3.19. By construction, the resolution Y is equipped with a \mathfrak{S}_3 -action. The stabilizer of any point for this action is trivial. Hence, passing to the quotient modulo \mathfrak{S}_3 , the induced map $\hat{f} : Y/\mathfrak{S}_3 \longrightarrow S^3 X$ provides an explicit resolution of $S^3 X$ which factors through the Hilbert scheme of points $X^{[3]} = B^3/\mathfrak{S}_3$.

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